On the role of conformal three-geometries in the dynamics of General Relativity

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It is shown that the Chern–Simons functional, built in the spinor representation from the initial data on spacelike hypersurfaces, is invariant with respect to infinitesimal conformal rescalings if and only if the vacuum Einstein equations are satisfied. As a consequence, we show that in the phase space the Hamiltonian constraint of vacuum general relativity is the Poisson bracket of the imaginary part of this Chern–Simons functional and Misner's time (essentially the 3-volume). Hence the vacuum Hamiltonian constraint is the condition on the canonical variables that the imaginary part of the Chern–Simons functional be constant along the volume flow. The vacuum momentum constraint can also be reformulated in a similar way as a (more complicated) condition on the change of the imaginary part of the Chern–Simons functional along the flow of York's time.

1. Introduction

In the initial value formulation of general relativity the evolution of states is described with respect to a *topological*, or coordinate time, and we can speak about the *metrical*, i.e. the physical time, e.g. the proper length of the history of a massive particle or of an observer, only *after* solving the evolution equations (see e.g. [1-3]). Although this does not yield any problem in classical physics, it yields serious conceptional difficulties in the quantum theory both of the matter fields on a curved spacetime and of the gravity itself (see e.g. [4-6]). The resolution of this difficulty could be the isolation of certain (matter and/or gravitational) degrees of freedom as the 'natural' time variable ('internal clock'), and the evolution of the remaining degrees of freedom would be described with respect to this 'internal time' variable [6-9]. However, this 'internal time' would be defined on the *phase space* of the physical system rather than in the spacetime ('external time'), and hence it is not a priori obvious that these two concepts of time should coincide even if we have a well defined external time. To illustrate the potential difficulties, let us consider the planar rotor, the simplest possible model of classical clocks: The position of the pointer is given by an angle coordinate $\varphi \in [0, 2\pi)$, and we say that the clock is running if $\dot\varphi$, the derivative of the 'internal time' φ with respect to the 'external time' t, is strictly positive (or negative). Then the actual 'internal time' shown by the clock is not only the value of the angle variable φ, but φ *plus* 2π*-times the number of full periods the clock has taken.* Thus φ could be a global 'internal time' only if $\dot\varphi$, as a function of φ, tends to zero as $\varphi \to 2\pi$, i.e. if the clock 'slows down to zero' asymptotically before taking one complete period. Thus if $\dot\varphi \geq c > 0$ for some constant c, then the 'internal time' φ cannot be globally well defined. In fact, by Poincare's recurrence theorem [10] this seems to be a general property of any *localized, quasi-stationary* clock modeled by a classical Hamiltonian mechanical system (even with noncompact configuration space): If the dynamics is forced to take place in an open subset W with compact closure of the phase space (e.g. by a potential increasing monotonically at infinity to bound the system's positions and momenta), then for any point $p \in W$ and its arbitrarily small open neighbourhood U there is an 'external time' parameter value $t_{(p,U)}$ such that the system's dynamical trajectory through p at $t = 0$ returns to U before $t_{(p,U)}$, and hence there is no continuous function on W

which would be monotonically increasing along the dynamical trajectory. (If the system is not localized, then, of course, one can find such a function, e.g. one of the Cartesian coordinates of a free particle.) Thus one should think of the 'external time', i.e. the natural parameter along the dynamical trajectories, only as an abstraction of the 'internal times' of specific localized clocks, like the manifold, which is defined in terms of local Euclidean neighbourhoods, and one cannot expect the 'internal times' to be global. What one can expect is that the *derivative* of one with respect to the other be bounded from below by a positive constant.

Returning to general relativity, technically the choice for such an 'internal clock' would be the fixing of the lapse function of the foliation in an intrinsic, geometric way. In fact, in the expanding or contracting phase of the Bianchi I. and IX. cosmological models Misner [11] found the volume of the hypersurface of the spatial homogeneity to be such a natural time coordinate. In the basic paper "Role of conformal three-geometries in the dynamics of general relativity" York showed how the unconstrained (physical) degrees of freedom of vacuum general relativity can be characterized by the conformal 3-geometry of the spacelike hypersurfaces, and one of the canonical momenta, the trace $T := \frac{2}{3\kappa}\chi$ of the extrinsic curvature of the spacelike hypersurfaces, could be identified as a natural time variable [12]. (Here κ is Einstein's gravitational constant.) In fact, by the Raychaudhury equation T is monotonic in time provided the strong energy condition holds (e.g. in vacuum) and the acceleration of the hypersurface can be neglected. Thus it may be monotonic even when Misner's time has a turning point. Unfortunately T is not monotonic in general either. Recently Smolin and Soo argued that since the proper arena of the dynamics is the phase space rather than the spacetime and in a canonical quantum theory the carrying space of the wave functions is the configuration space, we should find a natural time variable in the configuration space and not in the spacetime. For such a natural time variable in the configuration space they suggested the imaginary part of the Chern–Simons functional built from the complex Ashtekar connection [13]. Although the quantum dynamics must be formulated in the configuration space (or on the phase space endowed with an appropriate polarization), we think that in the *classical theory* time in the phase space or configuration space (i.e. the 'internal time') and time in the spacetime ('external time') should be monotonic with respect to each other, even if the former is not a globally defined observable on the phase- or configuration space. Unfortunately, their time variable is also of limited validity [14].

These negative results rise the question as whether this failure of finding the 'internal time' is an indication of the non-existence of such a time in the field theoretic framework either, at least in the generic case. It is an open question whether the dynamics of the vacuum Einstein equations are analogous to that of the free particle, or there exists an appropriate version of Poincare's recurrence theorem for infinite dimensional Hamiltonian systems with physically reasonable conditions that would rule out the existence of 'intrinsic times'. On the basis of a Machian analysis Barbour goes further [15] saying that the idea of the 'intrinsic time' and clocks is wrong, and "any satisfactory operational definition of time must involve all the degrees of freedom of the universe on an equal footing", and it is legitimate to use clocks *only* in the description of the dynamics of the *subsystems* of the whole universe.

The Chern–Simons functional is known to play an interesting role in the characterization of the conformal structures on 3-manifolds. In fact, the Chern–Simons functional built from the Levi-Civita connection of a Riemannian 3-manifold $(\Sigma, q_{\mu\nu})$ is invariant with respect to conformal rescalings of the 3-metric $q_{\mu\nu}$ [16]. Furthermore, in the Lichnerowitz–Choquet-Bruhat–York (conformal) approach of solving the constraint equations for the constant mean curvature data set of general relativity it is only the *conformal class* of the 3-metric, and the (unphysical) transverse–traceless extrinsic curvature and the trace of the physical extrinsic curvature that should be prescribed [17]. But the curvature of the conform 3-geometries is the Cotton–York tensor, which is just the variational derivative of the Chern–Simons functional above with respect to the metric [16]. Thus some role of the Chern–Simons functional in the dynamics of the 3+1 dimensional general relativity may be expected. In some of our previous papers we gave two generalizations of this Chern–Simons

conformal invariant for triples $(\Sigma, q_{\mu\nu}, \chi_{\mu\nu})$, where $\chi_{\mu\nu}$ is any symmetric tensor field on Σ [18]. The first was based on a real Lorentzian vector bundle over Σ and a covariant derivative thereon, built from $q_{\mu\nu}$ and $\chi_{\mu\nu}$. This is invariant with respect to transformations of $q_{\mu\nu}$ and $\chi_{\mu\nu}$ corresponding to *spacetime* conformal rescalings. The second generalization was based on the *complex* vector bundle of anti-self-dual 2-forms, but that is *not* invariant with respect to conformal rescalings. Later these two generalizations were shown to be special cases of a more general construction based on the bundle of spinors with k unprimed and l primed indices [14], and they can be recovered from that corresponding to the $(k,l) = (1,0)$ spinor representation and its complex conjugate. Thus it is enough to consider the Chern–Simons functional constructed in the basic spinor representation.

The present paper is addressed to the problem of dynamics of vacuum general relativity using certain three dimensional (conformal) geometries and the Chern–Simons functional, both in the spacetime and the phase space, but from a slightly different point of view. We consider a special spinorial 3-geometry, built from the initial data (and not only from the spatial 3-metric), and the Chern–Simons functional Y constructed in the basic spinor representation. We are arguing that the proper interpretation of the Misner and York times and the imaginary part of the spinor Chern–Simons functional is not 'internal time', rather they are observables on the phase space by means of which the constraints of vacuum general relativity can be rewritten into a new form. First we show that this Chern–Simons functional is invariant with respect to infinitesimal spacetime conformal rescalings if and only if the vacuum Einstein equations are satisfied. Then we reformulate this result in the ADM phase space of vacuum general relativity, and we show that this Chern–Simons functional generates the time evolution of vacuum general relativity in the sense that the Hamiltonian constraint is just the condition that the Poisson bracket of the imaginary part of Y and Misner's time be zero. (The Poisson bracket of $\mathrm{Re}\,Y$ and Misner's time is automatically zero.) The momentum constraint will also be reformulated as a condition on the Poisson bracket of $\mathrm{Im}\,Y$ and the integral of York's local time.

In Section 2. we introduce the Chern–Simons functional Y in a direct way, without referring to the Lorentzian vector bundle of [18,14], and review its most important properties that we need in what follows. In particular, we clarify the conformal properties of Y and show that it is invariant with respect to infinitesimal spacetime conformal rescalings iff the constraint parts of the vacuum Einstein equations are satisfied. In Section 3. the base manifold Σ is considered to be a spacelike hypersurface in the spacetime, and Y is shown to be invariant with respect to infinitesimal spacetime conformal rescalings on *every* spacelike hypersurface iff the vacuum Einstein equations (and not only their constraint parts) are satisfied. Then, in Section 4, the structures on the ADM phase space Γ_{ADM} that we need are reviewed, and, in particular, we introduce the Misner and York times and the spinor Chern–Simons functional as functions on Γ_{ADM}. In Section 5. the notion of conformal rescalings is implemented in Γ_{ADM}, and it is shown how the vacuum constraints of Einstein's theory can be reformulated by means of $\mathrm{Im}\,Y$. Finally, in Section 6, we discuss the Chern–Simons functional on the Ashtekar phase space.

Our conventions are mostly those of [19] (and follow [18,14]): In particular, the exterior product is defined as the anti-symmetrized tensor product and the spacetime signature is -2. The curvature and Ricci tensors e.g. of the covariant derivative operator ∇_a are defined by $-{}^4R^a{}_{bcd}W^bX^cY^d := \nabla_X\nabla_Y W^a - \nabla_Y\nabla_X W^a - \nabla_{[X,Y]}W^a$ and ${}^4R_{bd} := {}^4R^a{}_{bad}$, respectively, and the scalar curvature is ${}^4R := {}^4R_{ab}g^{ab}$. Thus, in the presence of matter, Einstein's equations take the form ${}^4G_{ab} := {}^4R_{ab} - \frac{1}{2}{}^4Rg_{ab} = -\kappa T_{ab}$, where $\kappa := 8\pi G$ and G is Newton's gravitational constant. Our differential geometric background is based primarily on [20].

2. The spinor Chern–Simons functional of the triples $(\Sigma, q_{\mu\nu}, \chi_{\mu\nu})$

Let Σ be a connected, orientable 3-manifold, $\mathbf{S}^A(\Sigma)$ a complex vector bundle of rank two over Σ and $\bar{\mathbf{S}}^{A'}(\Sigma)$ its complex conjugate bundle. Let ε_{AB} be a symplectic and $t_{AA'}$ a positive definite Hermitian fibre metric on $\mathbf{S}^A(\Sigma)$. We assume that ε_{AB} and $t_{AA'}$ are compatible in the sense that $2\varepsilon^{A'B'}t_{AA'}t_{BB'}=\varepsilon_{AB}$, and hence $\varepsilon^{AB}\varepsilon^{A'B'}t_{BB'}$ is just the inverse $t^{AA'}$ of $t_{AA'}$, where the inverses are defined by $\varepsilon^{AC}\varepsilon_{BC}:=\delta^A_B$ and $2t^{AA'}t_{BA'}:=\delta^A_B$, respectively. We identify $\mathbf{S}^A(\Sigma)$ with its dual $\mathbf{S}_A(\Sigma)$ via ε_{AB}, and $\bar{\mathbf{S}}^{A'}(\Sigma)$ with $\bar{\mathbf{S}}_{A'}(\Sigma)$ via $\varepsilon_{A'B'}$. An element $X^{AA'}$ of $\mathbf{S}^A(\Sigma)\otimes\bar{\mathbf{S}}^{A'}(\Sigma)$ is called real if $\bar{X}^{AA'}=X^{AA'}$, and note that $\varepsilon_{AB}\varepsilon_{A'B'}$ is a Lorentzian fibre metric with signature $(+,-,-,-)$ on the subbundle of the real elements of $\mathbf{S}^A(\Sigma)\otimes\bar{\mathbf{S}}^{A'}(\Sigma)$. $t_{AA'}$ can also be considered as a (real) element of $\mathbf{S}_A(\Sigma)\otimes\bar{\mathbf{S}}_{A'}(\Sigma)$, and it is timelike with unit length. $P^{AA'}_{BB'}:=\delta^A_B\delta^{A'}_{B'}-t^{AA'}t_{BB'}=\frac{1}{2}(\delta^A_B\delta^{A'}_{B'}-2t^A{}_{B'}t^{A'}{}_B)$ is the projection to the subbundle of the elements of $\mathbf{S}^A(\Sigma)\otimes\bar{\mathbf{S}}^{A'}(\Sigma)$ orthogonal to $t_{AA'}$, and hence any section $K^{AA'}$ of $\mathbf{S}^A(\Sigma)\otimes\bar{\mathbf{S}}^{A'}(\Sigma)$ has a unique decomposition into the sum of a section proportional to and orthogonal to $t^{AA'}$ as $K^{AA'}=Nt^{AA'}+N^{AA'}$. By a theorem of Stiefel every orientable 3-manifold is parallelizable, i.e. its tangent bundle is trivial (see e.g. [21]), thus if $\mathbf{S}^A(\Sigma)$ is chosen to be trivial then there is a globally defined (base point preserving) bundle injection $\Theta:T\Sigma\to\mathbf{S}^A(\Sigma)\otimes\bar{\mathbf{S}}^{A'}(\Sigma)\,:X^\mu\mapsto X^{AA'}:=X^\mu\Theta^{AA'}_\mu$, which is an isomorphism between the tangent bundle $T\Sigma$ of Σ and the bundle of the real elements of $\mathbf{S}^A(\Sigma)\otimes\bar{\mathbf{S}}^{A'}(\Sigma)$ orthogonal to $t_{AA'}$. By $\Theta^{AA'}_\mu t_{AA'}=0$ one has $\Theta^{AB}_\mu:=-\sqrt{2}\Theta^{AA'}_\mu t_{A'}{}^B=\Theta^{(AB)}_\mu$, and the pull back $q_{\mu\nu}:=\Theta^{AA'}_\mu\Theta^{BB'}_\nu\varepsilon_{AB}\varepsilon_{A'B'}$ of the Lorentzian fibre metric to $T\Sigma$ along Θ is real and negative definite. Thus the embedding $\Theta^{AA'}_\mu$ defines an $SU(2)$-spinor structure on Σ with $SU(2)$ soldering form Θ^{AB}_μ, which therefore satisfies $\Theta^A_{\mu B}\Theta^B_{\nu C}=\frac{\mathrm{i}}{\sqrt{2}}\varepsilon_{\mu\nu}{}^\rho\Theta^A_{\rho C}-\frac{1}{2}q_{\mu\nu}\delta^A_C$. Here $\varepsilon_{\mu\nu\rho}$ is the metric volume 3-form on Σ determined by $q_{\mu\nu}$. Although there might be other $SU(2)$-spinor structures on Σ, labeled by the elements of the cohomology group $H^1(\Sigma,\mathbf{Z}_2)$, by spinors on Σ we mean the elements of the trivial bundle $\mathbf{S}^A(\Sigma)$ above. By the triviality $\mathbf{S}^A(\Sigma)$ admits globally defined spin frame fields $\{\varepsilon^A_{\underline{A}}\}$, $\underline{A}=0,1$, normalized by $\varepsilon_{AB}\varepsilon^A_{\underline{A}}\varepsilon^B_{\underline{B}}=\epsilon_{\underline{A}\underline{B}}$, where $\epsilon_{\underline{A}\underline{B}}$ is the alternating Levi-Civita symbol. The dual spin frame field will be denoted by $\{\varepsilon^{\underline{A}}_A\}$. If the basis $\{\varepsilon^A_{\underline{A}}\}$ is normalized with respect to $t_{AA'}$, too, i.e. $t_{AA'}\varepsilon^A_{\underline{A}}\bar{\varepsilon}^{A'}_{\underline{A}'}=\sigma^0_{\underline{A}\underline{A}'}:=\frac{1}{\sqrt{2}}\mathrm{diag}(1,1)$, then it is called an $SU(2)$-spin frame. Let $\sigma^{\underline{a}}_{\underline{A}\underline{A}'}=(\sigma^0_{\underline{A}\underline{A}'},\sigma^{\mathbf{i}}_{\underline{A}\underline{A}'})$, $\underline{a}=0,...,3$ and $\mathbf{i}=1,2,3$, be the standard $SL(2,\mathbf{C})$ Pauli matrices (including the factor $1/\sqrt{2}$), and define the three 1-forms $\vartheta^{\mathbf{i}}_\mu:=\Theta^{AA'}_\mu\varepsilon^{\underline{A}}_A\bar{\varepsilon}^{\underline{A}'}_{A'}\sigma^{\mathbf{i}}_{\underline{A}\underline{A}'}$. It is easy to see that $\{\vartheta^{\mathbf{i}}_\mu\}$ is $q_{\mu\nu}$-orthonormal if $\{\varepsilon^A_{\underline{A}}\}$ is an $SU(2)$-spin frame: $\vartheta^{\mathbf{i}}_\mu\vartheta^{\mathbf{j}}_\nu\eta_{\mathbf{i}\mathbf{j}}=q_{\mu\nu}$, where $\eta_{\mathbf{i}\mathbf{j}}:=\mathrm{diag}(-1,-1,-1)$. Furthermore, if $\{\varepsilon^A_{\underline{A}}\}$ is a global spin frame field in $\mathbf{S}^A(\Sigma)$, then $\{\vartheta^{\mathbf{i}}_\mu\}$ is a global 1-form frame field in $T^*\Sigma$.

The metric $q_{\mu\nu}$ on $T\Sigma$ defines the Levi-Civita covariant derivative D_μ, whose action can be extended to $\mathbf{S}^A(\Sigma)$ by requiring $D_\mu\Theta^{AA'}_\nu=0$, $D_\mu\varepsilon_{AB}=0$ and $D_\mu t_{AA'}=0$. If $\{\varepsilon^A_{\underline{A}}\}$ is an $SU(2)$-spin frame, then the connection 1-form for D_μ on $\mathbf{S}^A(\Sigma)$ is well known to be expressible by the Ricci rotation coefficients defined in the basis $\{\vartheta^{\mathbf{i}}_\mu\}$. Explicitly, $\gamma^{\underline{A}}_{\mu\underline{B}}:=\varepsilon^{\underline{A}}_A D_\mu\varepsilon^A_{\underline{B}}=-\frac{\mathrm{i}}{2\sqrt{2}}(\vartheta^{\mathbf{i}}_\nu D_\mu e^\nu_{\mathbf{j}})\varepsilon_{\mathbf{i}}{}^{\mathbf{j}\mathbf{k}}\sigma^{\underline{A}}_{\mathbf{k}\underline{B}}$, where $\{e^\mu_{\mathbf{i}}\}$ is the vector basis in $T\Sigma$ dual to $\{\vartheta^{\mathbf{i}}_\mu\}$, $\varepsilon_{\mathbf{i}\mathbf{j}\mathbf{k}}$ is the alternating Levi-Civita symbol, and $\sigma^{\underline{A}}_{\mathbf{i}\underline{B}}:=\sqrt{2}\sigma^{\underline{A}}_{\mathbf{i}\underline{B}'}\epsilon^{\underline{B}'\underline{C}'}\sigma^0_{\underline{B}\underline{C}'}$, the standard $SU(2)$ Pauli matrices (including the coefficient $1/\sqrt{2}$). (Boldface indices are moved by $\eta_{\mathbf{i}\mathbf{j}}$ and its inverse. Thus $\varepsilon^{\mathbf{i}\mathbf{j}\mathbf{k}}$ is *minus* the Levi-Civita symbol $\epsilon^{\mathbf{i}\mathbf{j}\mathbf{k}}$.) Next we define another covariant derivative $\mathcal{D}_\mu$ on the spinor bundle $\mathbf{S}^A(\Sigma)$ by requiring that i. $\mathcal{D}_\mu\varepsilon_{AB}=0$, ii. $\chi_{\mu\nu}:=\chi_{\mu AA'}\Theta^{AA'}_\nu:=(\mathcal{D}_\mu t_{AA'})\Theta^{AA'}_\nu=\chi_{(\mu\nu)}$, and iii. $\mathcal{D}_\mu(v^\nu\Theta^{BB'}_\nu)P^{AA'}_{BB'}=(D_\mu v^\nu)\Theta^{AA'}_\nu$ for any vector field v^ν on Σ. Since $\mathcal{D}_\mu t^{AA'}$ and $\mathcal{D}_\mu(e^\nu_{\mathbf{i}}\Theta^{AA'}_\nu)$ are the $\mathcal{D}_\mu$-derivative of pointwise independent cross sections of $\mathbf{S}^A(\Sigma)\otimes\bar{\mathbf{S}}^{A'}(\Sigma)$, where $\{e^\nu_{\mathbf{i}}\}$ is any $q_{\mu\nu}$-orthonormal frame field, for given $\Theta^{AA'}_\mu$, $q_{\mu\nu}$ and $\chi_{\mu\nu}$ the conditions i.–iii. completely fix the derivative $\mathcal{D}_\mu$ on it. But since $\mathcal{D}_\mu$ annihilates not only the Lorentzian metric $\varepsilon_{AB}\varepsilon_{A'B'}$ on $\mathbf{S}^A(\Sigma)\otimes\bar{\mathbf{S}}^{A'}(\Sigma)$ but ε_{AB} itself, $\mathcal{D}_\mu$ is completely determined on $\mathbf{S}^A(\Sigma)$, too. Geometrically, $\chi_\mu{}^{AA'}$ measures the non-$t_{AA'}$-metricity of $\mathcal{D}_\mu$. To compare the derivatives $\mathcal{D}_\mu$ and D_μ, first let us observe that $\chi_\mu{}^{AB}:=-\sqrt{2}(\mathcal{D}_\mu t^{AA'})t_{A'}{}^B=\chi_\mu{}^{(AB)}$, and consider the $\mathcal{D}_\mu$-derivative of $K^{AA'}=Nt^{AA'}+N^{AA'}$. It is $\mathcal{D}_\mu K^{AA'}=(D_\mu N)t^{AA'}+N\mathcal{D}_\mu t^{AA'}+\mathcal{D}_\mu N^{AA'}=D_\mu K^{AA'}+(\chi_\mu{}^{AA'}t_{BB'}-t^{AA'}\chi_{\mu BB'})K^{BB'}$. Next, choosing $K^{AA'}$ to be $\lambda^A\bar{\mu}^{A'}$, this yields $\mathcal{D}_\mu\lambda^A=D_\mu\lambda^A-\frac{1}{\sqrt{2}}\chi_\mu{}^A{}_B\lambda^B$. Then the curvature of $\mathcal{D}_\mu$ can be computed easily: $F^A{}_{B\mu\nu}=R^A{}_{B\mu\nu}+$

$\frac{1}{\sqrt{2}}(D_\mu\chi_\nu{}^A{}_B - D_\nu\chi_\mu{}^A{}_B) - \frac{1}{2}(\chi_\mu{}^A{}_C\chi_\nu{}^C{}_B - \chi_\nu{}^A{}_C\chi_\mu{}^C{}_B)$, where $R^A{}_{B\mu\nu}$ is the curvature of D_μ on $\mathbf{S}^A(\Sigma)$. The connection 1-form and curvature 2-form of $\mathcal{D}_\mu$ in the global spin frame field above are $\Gamma^{\underline{A}}_{\mu\underline{B}} := \varepsilon^{\underline{A}}_A\mathcal{D}_\mu\varepsilon^A_{\underline{B}}$ and $F^{\underline{A}}{}_{\underline{B}\,\mu\nu} := \varepsilon^{\underline{A}}_A\varepsilon^B_{\underline{B}}F^A{}_{B\mu\nu}$, respectively. Since they are globally defined on Σ, we can form the Chern–Simons functional

$$Y[\Gamma^{\underline{A}}{}_{\underline{B}}] := \int_\Sigma \Bigl(F^{\underline{A}}{}_{\underline{B}\,\mu\nu}\Gamma^{\underline{B}}_{\rho\underline{A}} + \frac{2}{3}\Gamma^{\underline{A}}_{\mu\underline{B}}\Gamma^{\underline{B}}_{\nu\underline{C}}\Gamma^{\underline{C}}_{\rho\underline{A}}\Bigr)\frac{1}{3!}\delta^{\mu\nu\rho}_{\alpha\beta\gamma}, \tag{2.1}$$

provided the integral exists, e.g. if Σ is closed or if $(\Sigma, q_{\mu\nu}, \chi_{\mu\nu})$ is asymptotically flat in the sense that both $q_{\mu\nu}$ and $r\chi_{\mu\nu}$ tend to a flat metric and zero, respectively, at least logarithmically with the radial distance r [14]. $Y[\Gamma^{\underline{A}}{}_{\underline{B}}]$ is known to be invariant with respect to orientation preserving diffeomorphisms of Σ onto itself, and also with respect to basis transformations $\Lambda : \Sigma \to SL(2,\mathbf{C})$ of the spin frame that are homotopic to the identity transformation ("small gauge transformations"). However, under large (i.e. not small) gauge transformations $Y[\Gamma^{\underline{A}}{}_{\underline{B}}]$ changes by $16\pi^2N$, where N is integer and depends only on the homotopy class of Λ. In particular, the imaginary part of $Y[\Gamma^{\underline{A}}{}_{\underline{B}}]$ is gauge invariant. Since the quotient $SL(2,\mathbf{C})/SU(2)$ is homeomorphic to $\mathbf{R}^3$, furthermore any two continuous mappings $\lambda : \Sigma \to \mathbf{R}^3$ are homotopic, any global spin frame field can be transformed into an $SU(2)$-spin frame field by a small gauge transformation. Therefore, $Y[\Gamma^{\underline{A}}{}_{\underline{B}}]$ can always be computed in an $SU(2)$-spin frame, and hence it is determined completely by $\vartheta^{\mathbf{i}}_\mu$ and $\chi_{\mu\nu}$. Furthermore, since $Y[\Gamma^{\underline{A}}{}_{\underline{B}}]$ modulo $16\pi^2$ is gauge invariant, it is a functional only of $q_{\mu\nu}$ and $\chi_{\mu\nu}$ and will be denoted by $Y[q_{\mu\nu}, \chi_{\mu\nu}]$, or simply by Y. Its variational derivatives are

$$\begin{aligned}\frac{\delta Y}{\delta\chi_{\mu\nu}} =& 2\sqrt{|q|}H^{\mu\nu} + 2\mathrm{i}\sqrt{|q|}\Bigl(R^{\mu\nu} - \frac{1}{2}Rq^{\mu\nu} + V^{\mu\nu} - \frac{1}{2}Vq^{\mu\nu}\Bigr), &(2.2.a)\\ \frac{\delta Y}{\delta q_{\mu\nu}} =& -\sqrt{|q|}\Bigl(B^{\mu\nu} + \chi^{(\mu}{}_\rho H^{\nu)\rho}\Bigr) - \mathrm{i}\sqrt{|q|}\Bigl(\varepsilon^{\rho\omega(\mu}D_\rho H^{\nu)}{}_\omega + \frac{1}{2}D^{(\mu}\bigl(D_\rho\chi^{\nu)\rho} - D^{\nu)}\chi\bigr) - \\ &- \frac{1}{2}q^{\mu\nu}D_\omega\bigl(D_\rho\chi^{\rho\omega} - D^\omega\chi\bigr) + \chi^{(\mu}{}_\rho\bigl(R^{\nu)\rho} - \frac{1}{2}q^{\nu)\rho}R + V^{\nu)\rho} - \frac{1}{2}q^{\nu)\rho}V\bigr)\Bigr), &(2.2.b)\end{aligned}$$

where $R_{\mu\nu}$ is the Ricci tensor of the Riemannian 3-geometry $(\Sigma, q_{\mu\nu})$, R is its curvature scalar, we used the notations $V_{\mu\nu} := \chi\chi_{\mu\nu} - \chi_{\mu\rho}\chi^\rho{}_\nu$ and $V := V^\mu{}_\mu = \chi^2 - \chi^{\mu\nu}\chi_{\mu\nu}$, and introduced the tensor fields $H_{\mu\nu} := -\varepsilon_{\rho\omega(\mu}D^\rho\chi^\omega{}_{\nu)}$ and $B_{\mu\nu} := -\varepsilon_{\rho\omega(\mu}D^\rho R^\omega{}_{\nu)} - \varepsilon_{\rho\omega(\mu}D^\rho V^\omega{}_{\nu)} + \frac{1}{2}\chi^\rho{}_{(\mu}\varepsilon_{\nu)\rho\omega}D_\alpha(\chi^{\alpha\omega} - \chi q^{\alpha\omega})$. Note that both $H_{\mu\nu}$ and $B_{\mu\nu}$ are traceless and symmetric, and for vanishing $\chi_{\mu\nu}$ the latter reduces to the Cotton–York tensor of $(\Sigma, q_{\mu\nu})$. We have shown that the stationary points of $Y[q_{\mu\nu}, \chi_{\mu\nu}]$ are precisely those triples $(\Sigma, q_{\mu\nu}, \chi_{\mu\nu})$ that can be locally isometrically embedded into the Minkowski spacetime with first and second fundamental forms $q_{\mu\nu}$ and $\chi_{\mu\nu}$, respectively [18].

Let $\Omega : \Sigma \to (0,\infty)$, $\dot\Omega : \Sigma \to \mathbf{R}$ be smooth functions. The conformal rescaling of the metrics ε_{AB} and $t_{AA'}$ and the connection $\mathcal{D}_\mu$ by the pair $(\Omega, \dot\Omega)$ is defined by $\varepsilon_{AB} \mapsto \Omega\varepsilon_{AB}$, $t_{AA'} \mapsto \Omega t_{AA'}$ and $\chi_{\mu\nu} \mapsto \Omega\chi_{\mu\nu} + \dot\Omega q_{\mu\nu}$. This rescaling yields the change $\Gamma^{\underline{A}}_{\mu\underline{B}} \mapsto \tilde\Gamma^{\underline{A}}_{\mu\underline{B}} := \Gamma^{\underline{A}}_{\mu\underline{B}} - \frac{1}{\sqrt{2}}\Omega^{-1}(\delta^\rho_\mu\dot\Omega + \mathrm{i}\varepsilon_\mu{}^{\nu\rho}D_\nu\Omega)\Theta^{\underline{A}}_{\rho\underline{B}}$ of the connection 1-form, and, by a straightforward calculation, the following change of $Y[\Gamma^{\underline{A}}{}_{\underline{B}}]$:

$$\begin{aligned}Y[\tilde\Gamma^{\underline{A}}{}_{\underline{B}}] = Y[\Gamma^{\underline{A}}{}_{\underline{B}}] - 2\mathrm{i}\int_\Sigma \Bigl\{&\Omega^{-1}\Bigl(\frac{1}{2}(R+V)\dot\Omega + D_\mu\bigl(\chi^{\mu\nu} - q^{\mu\nu}\chi\bigr)(D_\nu\Omega)\Bigr) + \\ &+ \Omega^{-2}\Bigl(\chi\dot\Omega^2 + 2\dot\Omega q^{\mu\nu}D_\mu(D_\nu\Omega) + \chi^{\mu\nu}(D_\mu\Omega)(D_\nu\Omega)\Bigr) + \Omega^{-3}\Bigl(\dot\Omega^3 - \dot\Omega q^{\mu\nu}(D_\mu\Omega)(D_\nu\Omega)\Bigr)\Bigr\}\,\mathrm{d}\Sigma,\end{aligned} \tag{2.3}$$

where $\mathrm{d}\Sigma := \frac{1}{3!}\varepsilon_{\mu\nu\rho}$, the metric volume element. In particular, the real part of $Y[\Gamma^{\underline{A}}{}_{\underline{B}}]$ is invariant with respect to conformal rescalings. Let $(\Omega(u), \dot\Omega(u))$ be a smooth 1-parameter family of conformal factors such that $\Omega(0) = 1$ and $\dot\Omega(0) = 0$ (i.e. they represent the identity transformation at $u = 0$), and define

$\delta\Omega := (\mathrm{d}\Omega(u)/\mathrm{d}u)_{u=0}$ and $\delta\dot{\Omega} := (\mathrm{d}\dot{\Omega}(u)/\mathrm{d}u)_{u=0}$. By (2.3) under such an infinitesimal conformal rescaling $Y[\Gamma^{\underline{A}}{}_{\underline{B}}]$ transforms as $\delta Y[\Gamma^{\underline{A}}{}_{\underline{B}}] = -2\mathrm{i}\int_\Sigma(\frac{1}{2}(R+V)\delta\dot{\Omega} + D_\mu(\chi^{\mu\nu} - q^{\mu\nu}\chi)D_\nu\delta\Omega)\sqrt{|q|}\mathrm{d}^3x$. Remarkably enough, it is just the constraint parts of the spacetime Einstein tensor that appear in δY in spite of the fact that we have not assumed anything about the field equations, and *the spinor Chern–Simons functional is invariant with respect to infinitesimal conformal rescalings iff* $R + \chi^2 - \chi_{\mu\nu}\chi^{\mu\nu} = 0$ *and* $D_\mu(\chi^\mu{}_\nu - \delta^\mu_\nu\chi) = 0$.

The connection $\mathcal{D}_\mu$ on $\mathbf{S}^A(\Sigma)$ determines a unique connection on the bundle $\mathbf{S}^{(A_1\dots A_k)(B'_1\dots B'_l)}(\Sigma)$ of totally symmetric spinors on Σ with k unprimed and l primed indices. The Chern–Simons functional built from this connection, denoted by $Y_{(k,l)}$, has been shown to be determined completely by the spinor Chern–Simons functional and its complex conjugate [14]: $Y_{(k,l)} = \frac{1}{6}(k+1)(l+1)(k(k+2)Y[\Gamma^{\underline{A}}{}_{\underline{B}}] + l(l+2)\overline{Y[\Gamma^{\underline{A}}{}_{\underline{B}}]})$. In particular, the Chern–Simons functional in the (k,k) (real tensor) representations is proportional to the real part of $Y[\Gamma^{\underline{A}}{}_{\underline{B}}]$. The stationary points of $Y_{(k,k)}$, characterized by $H_{\mu\nu} = 0$ and $B_{\mu\nu} = 0$, are precisely those triples $(\Sigma, q_{\mu\nu}, \chi_{\mu\nu})$ that can be locally isometrically embedded into the conformal Minkowski spacetime with the first and second fundamental forms $q_{\mu\nu}$ and $\chi_{\mu\nu}$, respectively [18]. Furthermore, by (2.3) $Y_{(k,k)}$ is invariant with respect to conformal rescalings. In fact, $Y_{(k,k)}$ can be rewritten as the Chern–Simons functional built from the 3-surface local twistor connection on Σ, which is a manifestly conformally invariant expression [14]. Thus, in complete agreement with (2.3), it is the imaginary part of $Y[\Gamma^{\underline{A}}{}_{\underline{B}}]$ that in general breaks the conformal invariance.

3. The spinor Chern–Simons functional of the spacelike hypersurfaces

Let (M, g_{ab}) be a spacetime, $\theta_t : \Sigma \to M$, $t \in \mathbf{R}$, a foliation of M by spacelike hypersurfaces $\Sigma_t := \theta_t(\Sigma)$, and let t_a be the future directed unit timelike normal to the leaves Σ_t and $P^a_b := \delta^a_b - t^a t_b$ the corresponding projection to Σ_t. Since M is diffeomorphic to $\Sigma \times \mathbf{R}$, (M, g_{ab}) admits a spinor structure, and the spinor structures on (M, g_{ab}) are in a 1–1 correspondence with those on Σ. Thus there is one spinor bundle $\mathbf{S}^A(M)$ over M whose pull back to Σ is just the trivial $\mathbf{S}^A(\Sigma)$ above, and hence this $\mathbf{S}^A(M)$ is also globally trivializable. Let the corresponding $SL(2,\mathbf{C})$-soldering form on M be $\vartheta^{AA'}_a$. Then the bundle embedding $\Theta^{AA'}_\mu$ of the previous section can be identified with the composition of the differential $\theta_{t*}{}^a_\mu$ of the embedding θ_t and of the spacetime soldering form $\vartheta^{AA'}_a$. In particular, the spinor form $t^{AA'} := t^a\vartheta^{AA'}_a$ of the normal is just the positive definite Hermitian fibre metric, and the pull back $\theta_{t*}{}^a_\mu\mathcal{D}_a$ of the Sen connection $\mathcal{D}_a := P^b_a\nabla_b$ is just the covariant derivative $\mathcal{D}_\mu$ on $\mathbf{S}^A(\Sigma)$. $q_{\mu\nu}$ becomes the pull back to Σ of the induced metric $q_{ab} := P^c_a P^d_b g_{cd}$ on Σ_t, $\varepsilon_{\mu\nu\rho}$ the pull back of the induced volume form $\varepsilon_{abc} := \varepsilon_{abcd}t^d$, and $\chi_{\mu\nu}$ the pull back of the extrinsic curvature $\chi_{ab} := P^c_a P^d_b \nabla_c t_d$ of Σ_t in M with trace $\chi := \chi_{ab}q^{ab}$. (Our choice $\varepsilon_{abc} := \varepsilon_{abcd}t^d$ for the relation between the three and four dimensional volume forms is connected with the sign convention in the definition $\bar{\mathbf{S}}^{A'}(\Sigma) \to \mathbf{S}^X(\Sigma) : \bar{\lambda}^{A'} \mapsto -\sqrt{2}\bar{\lambda}^{A'}t_{A'}{}^X$ for the primed-unprimed correspondence of the contravariant spinor indices: The spinor form of the intrinsic volume 3-form is $\varepsilon_{\mu\nu\rho}\Theta^\mu_{AX}\Theta^\nu_{BY}\Theta^\rho_{CZ} = (\mathrm{i}/\sqrt{2})(\varepsilon_{A(B}\varepsilon_{Y)(C}\varepsilon_{Z)X} + \varepsilon_{X(B}\varepsilon_{Y)(C}\varepsilon_{Z)A})$, which coincides with the unitary spinor form of the induced volume 3-form only if the latter is defined by the convention above. Thus, although in differential geometry the volume form of a hypersurface is defined by the contraction of the normal with the *first* index of the volume form of the embedding geometry, we adopt the standard sign conventions in the theory of spinors rather than the standard sign conventions in differential geometry.) If ${}^4G_{ab}$ and ${}^4C_{abcd}$ are the spacetime Einstein and Weyl tensors, respectively, then ${}^4G_{ab}t^at^b = -\frac{1}{2}(R+V)$, ${}^4G_{ab}t^aP^b_c = -D_a(\chi^a{}_c - \delta^a_c\chi)$, and the conformal electric and magnetic curvatures are $E_{ab} := {}^4C_{acbd}t^bt^d = -(R_{ab} + V_{ab}) + \frac{1}{3}q_{ab}(R+V) + \frac{1}{2}P^c_aP^d_b({}^4G_{cd} - \frac{1}{3}q_{cd}q^{ef}\,{}^4G_{ef})$ and $H_{ab} := \frac{1}{2}\varepsilon_{ae}{}^{cd}\,{}^4C_{cdbf}t^et^f = -\varepsilon_{cd(a}D^c\chi^d{}_{b)}$, respectively. Thus although H_{ab} can be expressed by the induced metric and extrinsic curvature, in general E_{ab} cannot. The part of E_{ab} that is determined by the geometry of Σ_t is ${}_0E_{ab} := -(R_{ab} + V_{ab}) + \frac{1}{3}q_{ab}(R+V)$. The spacetime conformal rescaling $g_{ab} \mapsto \Omega^2 g_{ab}$ by

$\Omega : M \to (0,\infty)$ induces the transformations $q_{ab} \mapsto \Omega^2 q_{ab}$, $\chi_{ab} \mapsto \Omega\chi_{ab} + t^e(\nabla_e\Omega)q_{ab}$, which, identifying $\dot\Omega$ with $t^e\nabla_e\Omega$, justifies our definition for the conformal rescaling of $(\Sigma, \varepsilon_{AB}, t_{AA'}, \chi_{\mu\nu})$ in the previous section.

If $\{\varepsilon^A_{\underline{A}}\}$, $\{\varepsilon^{\underline{A}}_A\}$ is a (globally defined) dual spin frame field in M, then the connection 1-form ${}^4\Gamma^{\underline{A}}_{e\underline{B}} := \varepsilon^{\underline{A}}_A \nabla_e \varepsilon^A_{\underline{B}}$ and the curvature 2-form ${}^4R^{\underline{A}}{}_{\underline{B}\,cd} := \varepsilon^{\underline{A}}_A \varepsilon^B_{\underline{B}}\,{}^4R^A{}_{Bcd}$ are globally defined on M, and their pull back to Σ are just the connection and curvature forms, $\Gamma^{\underline{A}}_{\mu\underline{B}}(t)$ and $F^{\underline{A}}{}_{\underline{B}\,\mu\nu}(t)$, of the spinor connection on Σ, respectively. Thus the pull back to Σ of the 'spacetime' Chern–Simons 3-form ${}^4R^{\underline{A}}{}_{\underline{B}\,[ab}{}^4\Gamma^{\underline{B}}_{c]\underline{A}} + \frac{2}{3}{}^4\Gamma^{\underline{A}}_{[a|\underline{B}|}{}^4\Gamma^{\underline{B}}_{b|\underline{C}|}{}^4\Gamma^{\underline{C}}_{c]\underline{A}}$ is just the Chern–Simons 3-form built from the spinor connection $\Gamma^{\underline{A}}_{\mu\underline{B}}(t)$ on the bundle $\mathbf{S}^A(\Sigma)$. This 'spacetime' picture makes the conformal behaviour of $Y[\Gamma^{\underline{A}}{}_{\underline{B}}]$ more transparent. In fact, if $\Upsilon_{AA'} := \Omega^{-1}\nabla_{AA'}\Omega = \Omega^{-1}(\dot\Omega t_{AA'} + \Theta^\mu_{AA'}D_\mu\Omega)$, then

$$Y[\tilde\Gamma^{\underline{A}}{}_{\underline{B}}] = Y[\Gamma^{\underline{A}}{}_{\underline{B}}] - 2\mathrm{i}\int_{\Sigma_t}\Big\{-t_a\,{}^4G^{ab}\Upsilon_b + \mathrm{i}\varepsilon^{abcd}\Big(\big(\mathcal{D}_{AA'}\Upsilon_{B'C}\big)\Upsilon_{C'B} + \frac{2}{3}\Upsilon_{A'B}\Upsilon_{B'C}\Upsilon_{C'A}\Big)t_{DD'}\Big\}\mathrm{d}\Sigma_t. \quad (3.1)$$

Thus, as we saw in the previous section, the invariance of $Y[\Gamma^{\underline{A}}{}_{\underline{B}}]$ with respect to infinitesimal conformal rescalings is equivalent to ${}^4G_{ab}t^b = 0$ at arbitrary point p of a given hypersurface Σ_t, but boosting Σ_t slightly in three independent ways at p and repeating the previous argument, for the invariance of $Y[\Gamma^{\underline{A}}{}_{\underline{B}}]$ we get ${}^4G_{ab} = 0$. Therefore, *the spinor Chern–Simons functional is invariant with respect to infinitesimal conformal rescalings on every Cauchy surface iff the vacuum Einstein equations are satisfied.* Thus vacuum general relativity can be reformulated as an invariance requirement on the Chern–Simons functional built from a special spinor geometry on $\mathbf{S}^A(\Sigma)$ over the 3-manifold Σ. The obstruction to the invariance of Y is the spacetime Einstein tensor.

In this spacetime context we can calculate the 'time evolution' of $Y[\Gamma^{\underline{A}}{}_{\underline{B}}(t)]$, too. To do this first recall that the embedding $\theta_t : \Sigma \to M$ defines a congruence of curves on M by assigning the curve $\theta(t) := \theta_t(p)$ to the point $\theta_0(p) \in M$. Let us denote its tangent vector field by K^a, and decompose it into the sum of its parts normal and tangential to Σ_t: $K^a = Nt^a + N^a$. Then, using the identity $\text{Ł}_{\mathbf{K}} = \mathrm{d}\circ\iota_{\mathbf{K}} + \iota_{\mathbf{K}}\circ\mathrm{d}$ for the Lie derivative on the 'spacetime' Chern–Simons 3-form, for the time evolution we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}Y[\Gamma^{\underline{A}}{}_{\underline{B}}(t)] &= \int_{\Sigma_t}\frac{1}{2}\,{}^4R_{\underline{A}\,\underline{B}\,ab}\,{}^4R^{\underline{A}\,\underline{B}}{}_{cd}\varepsilon^{abcd}\,N\,\mathrm{d}\Sigma_t = \\ &= \int_{\Sigma_t}\Big\{4E_{ab}H^{ab} - \mathrm{i}\Big(2\big(E_{ab}E^{ab} - H_{ab}H^{ab}\big) - \frac{1}{2}{}^4G_{ab}\,{}^4G^{ab} + \frac{1}{6}\,{}^4R^2\Big)\Big\}N\mathrm{d}\Sigma_t,\end{aligned} \quad (3.2)$$

where we used the expressions for E_{ab} and H_{ab} above and those for the 'constraint parts' of the spacetime Einstein tensor by the three dimensional quantities. The real part of (3.2) is invariant with respect to spacetime conformal rescalings, thus the time derivative of the Chern–Simons functionals defined in the real tensor representations is also conformally invariant. In general neither the real nor the imaginary part has definite sign. Note that the shift vector does not appear on the right hand side of (3.2), showing the invariance of Y with respect to spatial diffeomorphisms. In the rest of the present paper we identify Σ with its image Σ_t, and hence use only the Latin indices.

4. The spinor Chern–Simon functional on the ADM phase space

The classical ADM phase space of vacuum general relativity (see, e.g. [1-3]), Γ_{ADM}, is the set of the pairs of fields $(q_{ab}, \tilde p^{ab})$ on a connected orientable 3-manifold Σ, where the configuration variables are the negative definite 3-metrics q_{ab} and the canonically conjugate momentum variables are the densitized symmetric

tensor fields $\tilde{p}^{ab}$. Thus the canonical symplectic 2-form ω is defined by $2\omega(\mathcal{X},\mathcal{X}') := \int_\Sigma((\delta\tilde{p}^{ab})(\delta' q_{ab}) - (\delta'\tilde{p}^{ab})(\delta q_{ab}))\mathrm{d}^3x$ for any tangent vectors $\mathcal{X} = (\delta q_{ab}, \delta\tilde{p}^{ab})$ and $\mathcal{X}' = (\delta' q_{ab}, \delta'\tilde{p}^{ab})$. In terms of the Cauchy data in a local coordinate system $\tilde{p}^{ab} = -\frac{1}{2\kappa}\sqrt{|q|}(\chi^{ab} - \chi q^{ab})$, where $q := \det(q_{ef})$, and χ_{ab} is the extrinsic curvature of Σ in the spacetime. Then the Hamiltonian and the momentum constraints of the vacuum general relativity are

$$\tilde{\mathcal{C}} := -\frac{1}{2\kappa}\sqrt{|q|}\Big(R - \frac{4\kappa^2}{|q|}\big(\tilde{p}^{ab}\tilde{p}^{cd}q_{ac}q_{bd} - \frac{1}{2}\big[\tilde{p}^{ab}q_{ab}\big]^2\big)\Big) = 0, \tag{4.1.a}$$

$$\tilde{\mathcal{C}}_a := 2q_{ab}D_c\tilde{p}^{bc} = 0. \tag{4.1.b}$$

The corresponding constraint functions on Γ_{ADM} are defined by $C[N, N^a] := \int_\Sigma(\tilde{\mathcal{C}}N + \tilde{\mathcal{C}}_a N^a)\mathrm{d}^3x$ for arbitrary function N and vector field N^a on Σ. If $\text{Ł}_{\mathbf{N}}$ denotes the Lie derivative operator along the vector field N^a, then their functional derivatives are

$$\begin{aligned}\frac{\delta C[N,N^a]}{\delta q_{ab}} &= \frac{1}{2\kappa}\sqrt{|q|}\Big\{N\Big(R^{ab} - Rq^{ab} + \frac{8\kappa^2}{|q|}\big(\tilde{p}^{ac}q_{cd}\tilde{p}^{bd} - \frac{1}{2}q_{cd}\tilde{p}^{cd}\tilde{p}^{ab}\big)\Big) + \\ &\qquad + D^aD^bN - q^{ab}D_eD^eN\Big\} - \frac{1}{2}N\tilde{\mathcal{C}}q^{ab} + \text{Ł}_{\mathbf{N}}\tilde{p}^{ab}, \qquad (4.2.a)\\ \frac{\delta C[N,N^a]}{\delta\tilde{p}^{ab}} &= \frac{4\kappa}{\sqrt{|q|}}N\Big(\tilde{p}_{ab} - \frac{1}{2}q_{ab}q_{cd}\tilde{p}^{cd}\Big) - \text{Ł}_{\mathbf{N}}q_{ab}. \qquad (4.2.b)\end{aligned}$$

It has already been shown that in the asymptotically flat case (with the standard $1/r$ fall-off and even parity for the metric and $1/r^2$ fall-off and odd parity for the canonical momenta) the vanishing of these functional derivatives and $C[N, N^a] = 0$ together imply that $N = 0$ and $N^a = 0$ [22], i.e. $C[N, N^a] = 0$ defines a non-degenerate 'surface' in Γ_{ADM}. However, in the closed case $C[N, N^a]$ does have critical points even if $C[N, N^a] = 0$. In fact, if $\text{Ł}_{\mathbf{N}}q_{ab} = 0$ (e.g. when N^a itself is vanishing), then by (4.2.b) $\tilde{p}^{ab} = 0$, and then by the Hamiltonian constraint $R = 0$. But by $\tilde{p}^{ab} = 0$ (4.2.a) takes the form $NR^{ab} + D^aD^bN - q^{ab}(D_eD^eN + \frac{1}{2}NR) = 0$, implying that $D_eD^eN = 0$ and $NR^{ab} = -D^aD^bN$. But the first, together with the compactness of Σ, implies that $N =$ const, and hence the second implies that q_{ab} is flat. Therefore, for flat q_{ab} the pair $(q_{ab}, 0) \in \Gamma_{ADM}$ is a critical point of the constraint function $C[N, 0]$ for constant N. Since, however, the shift vector is a part of the spacetime diffeo gauge freedom, $C[N, N^a]$ is expected to have critical points representing the flat spacetime. This result is in complete agreement with the classical result [23] that the closed flat spacetimes are unstable in the sense that not all solutions of the linearized constraints correspond to nearby solutions of the constraint equations themselves.*

Recall that a vector field $\mathcal{X}$ on Γ_{ADM} is called the Hamiltonian vector field of the function $\Phi : \Gamma_{ADM} \to \mathbf{R}$ if $2\omega(\mathcal{X},\mathcal{Y}) + \mathcal{Y}(\Phi) = 0$ for every vector $\mathcal{Y}$. The vector field $\mathcal{X}_\Phi := (\delta\Phi/\delta\tilde{p}^{ab}, -\delta\Phi/\delta q_{ab})$ is a Hamiltonian vector field of Φ, and, in fact, for finite dimensional symplectic manifolds the analogous expression follows from the definition of the Hamiltonian vector fields. In infinite dimensional phase spaces, however, the non-degeneracy of ω itself does not imply its invertability. Γ_{ADM} would have to be endowed with a reflexive Banach manifold structure. Thus, in lack of additional assumptions on ω, the definition of the Hamiltonian vector fields above does not imply this explicit expression for $\mathcal{X}_\Phi$ [24]. Thus we call $\mathcal{X}_\Phi$ the Hamiltonian vector field of Φ in the strong sense. In particular, the Hamiltonian vector field of $C[N, N^a]$ in the strong sense is $\mathcal{X}_{C[N,N^e]} = (\delta C[N, N^e]/\delta\tilde{p}^{ab}, -\delta C[N, N^e]/\delta q_{ab})$. The flow on Γ_{ADM} corresponding to $C[0, N^a]$ is the system of equations $\dot{q}_{ab} = -\text{Ł}_{\mathbf{N}}q_{ab}$, $\dot{\tilde{p}}^{ab} = -\text{Ł}_{\mathbf{N}}\tilde{p}^{ab}$; i.e. it is the natural lift of the vector field $-N^a$

* I am grateful to Niall Ó Murchadha for this remark and for pointing out reference [23].

to the phase space. Hence $C[0,N^a]$ generates a spatial diffeomorphism on Σ. The flow corresponding to $C[N,0]$ is just the system of evolution equations of the initial value formulation of the vacuum general relativity with vanishing shift vector. The Poisson bracket of the constraint functions is well known to be $\{C[N,N^a],C[M,M^a]\}=C[\text{Ł}_{\mathbf{N}}M-\text{Ł}_{\mathbf{M}}N,[N,M]^a+MD^aN-ND^aM]$. Thus $N^a\mapsto C[0,N^a]$ defines a Lie algebra homomorphism of the Lie algebra of vector fields $\mathrm{Vect}(\Sigma)$ into the Poisson algebra of functions $C^\infty(\Gamma_{ADM},\mathbf{R})$. For spatially closed spacetimes, which we are concentrating on for the sake of simplicity, the Hamiltonian is just the constraint with arbitrary lapse and shift: $H[N,N^a]:=-C[N,N^a]$. Therefore, in the Hamiltonian the two constraints play different roles: while $C[N,0]$ generates the proper evolution of the states with respect to the coordinate time, i.e. the *dynamics*, $C[0,N^a]$ generates only a smooth *kinematical symmetry* of the theory, i.e. it ensures the invariance of the theory with respect to *spatial* diffeomorphisms $\Sigma\to\Sigma$ that are homotopic to the identity mapping of Σ onto itself ("small diffeomorphisms') [1-3]. (For a different interpretation of the constraints see [15].)

Next let us define $V[n]:=\int_\Sigma n\sqrt{|q|}\mathrm{d}^3x$ and $T[f]:=\frac{2}{3}\int_\Sigma fq_{ab}\tilde p^{ab}\mathrm{d}^3x$ for any fixed $n,f:\Sigma\to\mathbf{R}$. If n is chosen to be the characteristic function of a subset $D\subset\Sigma$ then $V[n]$ becomes the metric volume $\mathrm{Vol}(D)$ of D, and $T[f]$ is the integral of York's local time smeared by f. (The area of a smooth orientable 2-surface $\mathcal{S}$ can also be recovered in a similar way.) Their Poisson bracket is $\{T[f],V[n]\}=V[fn]$, i.e. $T[1]$ acts on $V[n]$ as identity and hence $V[n]$ changes exponentially along the integral curves of the Hamiltonian vector fields of $T[1]$: If we write $V[n]=:V_0\exp(v[n])$ for some constant V_0, then we have $\{T[1],v[n]\}=1$. Misner's time is $-\frac{1}{3}v[n]$. Neither $V[n]$ nor $T[f]$ has critical points on Γ_{ADM}. Their time evolution is:

$$\dot V[n]:=\Big\{H[N,N^a],V[n]\Big\}=-V[\text{Ł}_{\mathbf{N}}n]+\frac{3\kappa}{2}T[Nn], \tag{4.3.a}$$

$$\dot T[f]:=\Big\{H[N,N^a],T[f]\Big\}=-C[fN,0]-T[\text{Ł}_{\mathbf{N}}f]-\frac{2}{3\kappa}V[f(D_eD^eN+RN)]. \tag{4.3.b}$$

Since in general $T[f]$ does not have any definite sign, $V[n]$, i.e. Misner's time, can in fact be monotonic only in rather special situations. Similarly, if f is constant and the Hamiltonian constraint is satisfied, then $T[f]$ is monotonic only for those *specific* lapses N for which the integral of RN on Σ is positive or negative. Such lapses always exist if R is not identically vanishing, ensuring the monotonity of $T[f]$, at least in a small coordinate time interval.

Since the spinor Chern–Simons functional $Y[\Gamma^{\underline{A}}{}_{\underline{B}}]$ modulo $16\pi^2$ is a well defined function of q_{ab} and χ_{ab}, it defines a function $Y=Y[q_{ab},\tilde p^{ab}]$ on the ADM phase space as well. Note that Y does not depend on any smearing function, furthermore, it is a dimensionless quantity. Its variational derivatives with respect to the ADM variables can be calculated using (2.2). They are

$$\begin{aligned}\frac{\delta Y}{\delta q_{ab}}=&\kappa\Big(-6\tilde p^{(a}{}_c\big(H^{b)c}-\mathrm{i}\,{}_0E^{b)c}\big)+2q^{ab}\tilde p^{cd}\big(H_{cd}-\mathrm{i}\,{}_0E_{cd}\big)+\tilde p\big(H^{ab}-\mathrm{i}\,{}_0E^{ab}\big)\Big)-\\&-\mathrm{i}\sqrt{|q|}\varepsilon^{cd(a}D_c\big(H^{b)}{}_d-\mathrm{i}\,{}_0E^{b)}{}_d\big)+\frac{\mathrm{i}}{2}\kappa\Big(D^{(a}\tilde{\mathcal{C}}^{b)}-q^{ab}D_c\tilde{\mathcal{C}}^c\Big)-\frac{\kappa^2}{\sqrt{|q|}}\tilde p^{(a}{}_c\varepsilon^{b)cd}\tilde{\mathcal{C}}_d,\end{aligned} \tag{4.4.a}$$

$$\frac{\delta Y}{\delta\tilde p^{ab}}=-4\kappa\Big(H_{ab}-\mathrm{i}_0E_{ab}\Big)+\frac{2}{3}\kappa^2\frac{\mathrm{i}}{\sqrt{|q|}}\tilde{\mathcal{C}}q_{ab}. \tag{4.4.b}$$

By (4.4) the critical points of Y are those for which ${}_0E_{ab}=0$, $H_{ab}=0$, $\tilde{\mathcal{C}}=0$, $D_{(a}\tilde{\mathcal{C}}_{b)}=0$ and $\tilde p^{(a}{}_c\varepsilon^{b)cd}\tilde{\mathcal{C}}_d=0$. As we have already shown [18], ${}_0E_{ab}=0$, $H_{ab}=0$ and $D_{(a}\tilde{\mathcal{C}}_{b)}=0$ imply that $\tilde{\mathcal{C}}_a=0$. Thus, in particular, the critical points of Y are all on the constraint surface, and, as we noted in Section 1, they represent initial data for locally flat spacetimes. The time evolution of Y, defined by $\dot Y:=\{H[N,N^a],Y\}=-\{C[N,N^a],Y\}$,

is just (3.2), where ${}^4G_{ab}=0$ (and hence ${}^4R=0$ and $E_{ab}={}_0E_{ab}$). Thus, in particular, the Poisson bracket of Y with the momentum constraint is zero, expressing its invariance with respect to small diffeomorphisms in the symplectic framework. However, apart from special configurations describing e.g. Petrov III. or N. spacetimes, the Poisson bracket with the Hamiltonian constraint is non-zero, and, apart from exceptional cases again, its sign is not definite. Thus the Hamiltonian vector field of neither $\mathrm{Re}\,Y$ nor $\mathrm{Im}\,Y$ is tangent to the constraint surface in Γ_{ADM}, and neither $\mathrm{Re}\,Y$ nor $\mathrm{Im}\,Y$ is monotonic during the time evolution. On the other hand, for *specific* lapses $\mathrm{Im}\,\dot Y$ can be ensured to be positive, and hence, in a small coordinate time interval, $\mathrm{Im}\,Y$ can be used as an 'internal time function'.

To summarize, these specific candidates for the 'internal time function' have two main drawbacks: First, they are monotonic only for *specific* lapse functions instead of any N, and, second, they are not globally defined. In fact, the first implies the second: Since the lapse must be chosen to be 'adapted' to the initial state $(q_{ab},\tilde p^{ab})$ to ensure the positivity of the time derivative of the 'time function', this 'adaptation' may go wrong during the evolution of the state. Thus a good natural time variable would have to be a function $\tau:\Gamma_{ADM}\to\mathbf{R}$ whose derivative $\dot\tau:=\{H[N,N^a],\tau\}$ is positive *for any positive lapse* N. Such a derivative could be, for example, $V[n]$ with some non-negative smearing function n, or the Bel–Robinson 'energy' $E_{BR}[n]:=\int_\Sigma({}_0E_{ab}\,{}_0E^{ab}+H_{ab}\,H^{ab})n\sqrt{|q|}\mathrm{d}^3x$ also with non-negative n. (The latter could be more natural because the vanishing of $E_{BR}[n]$ for any non-negative n, together with the vacuum constraints, implies that the corresponding initial data is flat, i.e. precisely the critical points of the vacuum constraints. It might be interesting to note that, apart from the sign in front of ${}_0E_{ab}\,{}_0E^{ab}$ in $E_{BR}[n]$, the imaginary part of (3.2) in vacuum is just this Bel–Robinson 'energy'.) However, the question of the globality of τ would still be open, as it is not clear e.g. how the infinite dimensional versions of Poincare's recurrence theorem restrict the possibility of globally defined time functions, like in the phase space of mechanical systems. We will see in the next two sections that $V[n]$ plays, in fact, the role of determining the *scale* of a time parameter (rather than the time itself), but in a slightly different context.

5. The role of spinor Chern–Simon functional in the dynamics of GR

The conformal rescaling of (q_{ab},χ_{ab}) yields the mapping $(q_{ab},\tilde p^{ab})\mapsto(\Omega^2q_{ab},\tilde p^{ab}+\frac{1}{\kappa}\Omega^{-1}\dot\Omega\sqrt{|q|}q^{ab})$ of the phase space onto itself. Thus the infinitesimal conformal rescaling, characterized by the pair of functions $(\delta\Omega,\delta\dot\Omega)$, defines the vector field $\mathcal{K}:=(\delta q_{ab},\delta\tilde p^{ab})=(2\delta\Omega q_{ab},\frac{1}{\kappa}\delta\dot\Omega\sqrt{|q|}q^{ab})$ on Γ_{ADM}. Then the effect of the infinitesimal conformal rescalings on a functionally differentiable function $F:\Gamma_{ADM}\to\mathbf{C}$ is $\mathcal{K}(F):=\int_\Sigma(\frac{\delta F}{\delta q_{ab}}\delta q_{ab}+\frac{\delta F}{\delta\tilde p^{ab}}\delta\tilde p^{ab})\mathrm{d}^3x$. In particular, this action on the functions $V[n]$, $T[f]$ and Y, respectively, is $\mathcal{K}(V[n])=3V[n\delta\Omega]$, $\mathcal{K}(T[f])=2T[f\delta\Omega]+\frac{2}{\kappa}V[f\delta\dot\Omega]$ and $\mathcal{K}(Y)=2\mathrm{i}\kappa\,C[\delta\dot\Omega,D^a\delta\Omega]$. The latter is simply a reformulation of (the u-derivative of) (2.3) in the symplectic framework, and hence the vacuum constraints are equivalent to the invariance requirement $\mathcal{K}(Y)=0$ for any $\mathcal{K}$ above.

Next let us clarify whether $\mathcal{K}$ is a Hamiltonian vector field or not. Suppose that $\mathcal{K}$ is a Hamiltonian vector field of a function Φ in the strong sense, and hence Φ satisfies

$$\frac{\delta\Phi}{\delta\tilde p^{ab}}=2q_{ab}\delta\Omega,\qquad\frac{\delta\Phi}{\delta q_{ab}}=-\frac{1}{\kappa}\sqrt{|q|}q^{ab}\delta\dot\Omega.\tag{5.1}$$

Then let us consider a smooth two-parameter family of points $(q_{ab}(u,v),\tilde p^{ab}(u,v))$ of Γ_{ADM}, $u,v\in(-\epsilon,\epsilon)$, and consider the function Φ as a function of the two parameters: $\Phi=\Phi(u,v)$. Using (5.1), the derivative of $\Phi(u,v)$ with respect to u at $u=0$, which is still a function of v, is

$$\delta_u\Phi := \Big(\frac{\mathrm{d}}{\mathrm{d}u}\Phi(u,v)\Big)_{u=0} = \int_\Sigma\Big\{\frac{\delta\Phi}{\delta q_{ab}}\delta_u q_{ab} + \frac{\delta\Phi}{\delta\tilde p^{ab}}\delta_u\tilde p^{ab}\Big\}\mathrm{d}^3x =$$
$$= \int_\Sigma\Big\{-\frac{1}{\kappa}\delta\dot\Omega q^{ab}\sqrt{|q|}\big(\delta_u q_{ab}\big) + 2\delta\Omega q_{ab}\big(\delta_u\tilde p^{ab}\big)\Big\}\mathrm{d}^3x,$$

and there is a similar expression for the v-derivative $\delta_v\Phi$, too. Since, however, $\Phi(u,v)$ is a smooth real valued function of two variables, the order of its u and v derivatives can be interchanged ('functional integrability condition'). Therefore,

$$0 = \delta_v\delta_u\Phi - \delta_u\delta_v\Phi = 2\int_\Sigma \delta\Omega\Big(\big(\delta_u\tilde p^{ab}\big)\big(\delta_v q_{ab}\big) - \big(\delta_v\tilde p^{ab}\big)\big(\delta_u q_{ab}\big)\Big)\mathrm{d}^3x.$$

However, apart from $\delta\Omega$, the right hand side is just the canonical symplectic 2-form evaluated on the vectors $(\delta_u q_{ab}, \delta_u\tilde p^{ab})$ and $(\delta_v q_{ab}, \delta_v\tilde p^{ab})$, which is non-vanishing. Therefore, Φ can be a non-trivial solution of (5.1) only if $\delta\Omega = 0$, and hence $\mathcal{K}$ can be a Hamiltonian vector field (in the strong sense) only for the special infinitesimal conformal rescalings with $\delta\Omega = 0$. Thus the spatial conformal rescalings are not canonical transformations, but the temporal ones, characterized by $\delta\dot\Omega$, are. In this special case (5.1) can be integrated immediately: $\Phi = -\frac{2}{\kappa}V[\delta\dot\Omega]$. This and the expression for $\mathcal{K}(Y)$ above lead to consider the Poisson bracket of $V[n]$ and Y. It is $\{Y, V[n]\} = \mathrm{i}\kappa^2\, C[n,0]$, i.e.

$$\Big\{\mathrm{Re}\, Y, V[n]\Big\} = 0, \qquad \Big\{\mathrm{Im}\, Y, V[n]\Big\} = \kappa^2\; C[n,0]. \tag{5.2}$$

Therefore, *the Hamiltonian constraint function $C[N,0]$ is a pure Poisson bracket of two naturally defined functions, and the geometric content of the Hamiltonian constraint is that the Hamiltonian vector field of the imaginary part of the spinor Chern–Simons functional be volume preserving, or, equivalently,* $\mathrm{Im}\, Y$ *must be constant along the flow of the Hamiltonian vector field of $V[N]$.* Thus, as Smolin and Soo have already realized, the imaginary part of Y should be connected with the time evolution, but its proper interpretation is not an 'intrinsic time function', rather it is a more elementary functional by means of which the constraint governing the dynamics of vacuum general relativity is generated by $V[N]$. Thus the time evolution is governed by the integral of a potential for a topological quantity (viz. the second Chern class of the spacetime) and the 3-volume. The lapse function enters $C[N,0]$ only through $V[N]$, while the Chern–Simons functional, being dimensionless and depending on no smearing function, appears to be some 'universal master function'. $V[N]$ is not a time function either, rather it is a 'sub-generator' of the dynamics of the vacuum general relativity, and, through the Poisson bracket with $\mathrm{Im}\, Y$, it determines the scale of the natural parameter of the integral curves of the Hamiltonian vector field of $C[N,0]$. The Hamiltonian vector field of the real part of Y is automatically volume preserving, a manifestation of its conformal invariance in the symplectic framework. Remarkably enough, the 3-volume has already appeared in connection with the dynamics of general relativity: First, as Misner's time [11], or recently in the reduced Hamiltonian of Fischer and Moncrief [25] (see also [6,26]). By (5.2) it is perhaps more natural to interpret $C[N,0] = 0$ as the condition that Y must be constant along the flow of $V[N]$, because $V[N]$ does not have critical points.

Although for nonzero $\delta\Omega$ there is no differentiable function Φ on the phase space which would be a solution of (5.1) (i.e. there is no function Φ for which $2\omega(\mathcal{K},\mathcal{X}) + \mathcal{X}(\Phi) = 0$ would hold for *any* vector $\mathcal{X}$), for *specific* $\mathcal{X}$, namely for the Hamiltonian vector field $\mathcal{X}_{\mathrm{Im}\, Y}$ of $\mathrm{Im}\, Y$, there may exist a function Φ for which $2\omega(\mathcal{K},\mathcal{X}_{\mathrm{Im}\, Y}) + \mathcal{X}_{\mathrm{Im}\, Y}(\Phi) = 0$ could hold. Or, in other words, the momentum constraint $C[0,N^a]$ may still be expected to be the Poisson bracket of $\mathrm{Im}\, Y$ and some real $W[N^a]$, or to be $\mathrm{Im}\,\{Y, W\}$ for some complex W. However, contrary to expectations, the Poisson bracket of Y with $T[f]$ is not only the momentum constraint. It is

$$
\begin{aligned}
3\Big\{\mathrm{Re}\,Y, T[f]\Big\} &= -8\kappa\int_\Sigma f\tilde p^{ab}H_{ab}\mathrm{d}^3x = -\frac{16\kappa^2}{3}\int_\Sigma\frac{f}{\sqrt{|q|}}\varepsilon^{cda}\big(D_c\tilde p^b{}_d\big)\tilde p_{ab}\,\mathrm{d}^3x,\\
3\Big\{\mathrm{Im}\,Y, T[f]\Big\} &= 8\kappa\int_\Sigma f\tilde p^{ab}\,{}_0E_{ab}\mathrm{d}^3x + \frac{4\kappa^2}{3}\int_\Sigma\frac{f}{\sqrt{|q|}}\tilde p\,\tilde{\mathcal C}\,\mathrm{d}^3x - 2\kappa\,C[0, D^af].
\end{aligned}
\tag{5.3}
$$

Therefore, *the constraints of the vacuum Einstein theory* $C[N, N^a] = 0$ *are equivalent to the conditions* $\{\mathrm{Im}\,Y, V[N]\} = 0$ *and* $3\{\mathrm{Im}\,Y, T[f]\} = 8\kappa\int_\Sigma f\tilde p^{ab}\,{}_0E_{ab}\mathrm{d}^3x$ *on the Hamiltonian vector field of* $\mathrm{Im}\,Y$*, i.e.* $\mathrm{Im}Y$ *is constant along the volume flow, and varies in a specific way along the flow of* $T[f]$. It is not clear whether the momentum constraint function can also be written as the Poisson bracket of Y (or of $\mathrm{Im}\,Y$) and some other function on Γ_{ADM} (whenever the momentum constraint could also be interpreted as the condition of the invariance of Y (or of $\mathrm{Im}\,Y$) along the flow of another Hamiltonian vector field), or not.

6. The spinor Chern–Simons functional on the Ashtekar phase space

By the triviality of $\mathbf{S}^A(\Sigma)$ the bundle of symmetric unprimed spinors is also globally trivializable: $\mathbf{S}^{(AB)}(\Sigma) \approx \Sigma\times\mathbf{C}^3$, and if $\{\varepsilon^A_{\underline A}\}$ is a global spin frame field in $\mathbf{S}^A(\Sigma)$, then $\varepsilon^{AB}_{\mathbf i} := \sigma^{\underline A\,\underline B}_{\mathbf i}\varepsilon^A_{\underline A}\varepsilon^B_{\underline B}$, $\mathbf{i} = 1,2,3$, is a global frame field in $\mathbf{S}^{(AB)}(\Sigma)$ and orthonormal with respect to the natural scalar product $\langle w^{AB}, z^{AB}\rangle := w^{AB}z^{CD}\varepsilon_{AC}\varepsilon_{BD}$. For fixed bundle injection $\Theta^{AA'}_a$ the $SU(2)$ soldering form Θ^{AB}_a defines a (non-canonical) bundle isomorphism $\Theta: T\Sigma\otimes\mathbf{C}\to\mathbf{S}^{(AB)}(\Sigma)$, and, in particular, the global frame field $\varepsilon^{AB}_{\mathbf i}$ can be written as $\varepsilon^{AB}_{\mathbf i} = E^a_{\mathbf i}\Theta^{AB}_a$ for some globally defined complex basis $\{E^a_{\mathbf i}\}$ in $T\Sigma\otimes\mathbf{C}$. By the definitions, $\{E^a_{\mathbf i}\}$ is orthonormal with respect to q_{ab}. However, the basis $\{E^a_{\mathbf i}\}$ should not be confused with the orthonormal basis $\{e^a_{\mathbf i}\}$ of $T\Sigma$ used in Section 2. The former is in fact a (complex) basis of $\mathbf{S}^{(AB)}(\Sigma)$ in a disguise. Its densitized form, defined in a local coordinate system by $\tilde E^a_{\mathbf i} := \sqrt{|q|}E^a_{\mathbf i}$, is therefore a triad of complex vectors of weight one. The connection 1-form and its curvature on $\mathbf{S}^{(AB)}(\Sigma)$ in this basis can be represented by $A^{\mathbf i}_a := \frac12\varepsilon^{\mathbf i}{}_{\mathbf{jk}}A^{\mathbf{jk}}_a := \mathrm{i}\sqrt2\sigma^{\mathbf i}_{\underline A\,\underline B}\Gamma^{\underline A\underline B}_a$ and $F^{\mathbf i}{}_{ab} := \frac12\varepsilon^{\mathbf i}{}_{\mathbf{jk}}F^{\mathbf{jk}}{}_{ab} := \mathrm{i}\sqrt2\sigma^{\mathbf i}_{\underline A\,\underline B}F^{\underline A\,\underline B}{}_{ab}$, respectively. The latter can also be given by $F_{\mathbf{ij}} := F_{\mathbf i\,ab}E^a_{\mathbf k}E^b_{\mathbf l}\varepsilon^{\mathbf{kl}}{}_{\mathbf j}$, whose complete irreducible decomposition into its trace, anti-symmetric- and trace–free symmetric parts, expressed by the initial data of the second and third sections, is

$$
\begin{aligned}
F_{\mathbf{ij}} &= -\frac13F^{\mathbf{kl}}{}_{ab}E^a_{\mathbf k}E^b_{\mathbf l}\eta_{\mathbf{ij}} - F^{\mathbf l}{}_{ab}E^a_{\mathbf l}E^b_{\mathbf k}\varepsilon^{\mathbf k}{}_{\mathbf{ij}} + F_{\langle\mathbf{ij}\rangle} =\\
&= -\frac13\big(R+V\big)\eta_{\mathbf{ij}} - \mathrm{i}D_b\big(\chi^b{}_a - \delta^b_a\chi\big)E^a_{\mathbf k}\varepsilon^{\mathbf k}{}_{\mathbf{ij}} - 2\big({}_0E_{ab} + \mathrm{i}H_{ab}\big)E^a_{\mathbf i}E^b_{\mathbf j}.
\end{aligned}
\tag{6.1}
$$

Note that for given $\tilde E^a_{\mathbf i}$ and $A^{\mathbf i}_a$ the ADM canonical variables q_{ab} and $\tilde p^{ab}$ are uniquely determined.

The Ashtekar phase space Γ_A is defined to be the set of the pairs $(A^{\mathbf i}_a, \tilde E^a_{\mathbf i})$ of the $so(3,\mathbf{C})$-valued connection forms and the complex triads of weight one, endowed with the symplectic structure whose restriction to the domain of the Ashtekar map $\mathtt{A}: \Gamma_A\to\Gamma_{ADM}: (A^{\mathbf i}_a, \tilde E^a_{\mathbf i})\mapsto(q_{ab}, \tilde p^{ab})$ above is just the symplectic structure pulled back from Γ_{ADM} along $\mathtt{A}$. This symplectic structure is $\mathrm{i}\kappa$-times the natural symplectic structure of Γ_A: If $A[\tilde\omega] := \int_\Sigma A^{\mathbf i}_a\tilde\omega^a_{\mathbf i}\mathrm{d}^3x$ and $E[\varphi] := \int_\Sigma\tilde E^a_{\mathbf i}\varphi^{\mathbf i}_a\mathrm{d}^3x$, the basic field variables smeared by arbitrary test fields with appropriate weights, then their Poisson bracket is $\{E[\varphi], A[\tilde\omega]\} = \mathrm{i}\kappa\int_\Sigma\varphi^{\mathbf i}_a\tilde\omega^a_{\mathbf i}\mathrm{d}^3x$. As a consequence of the non-injectivity of $\mathtt{A}$, i.e. the extra (internal) gauge freedom coming from the use of the triads $\tilde E^a_{\mathbf i}$ instead of the metrics q_{ab}, a further constraint, the so-called Gauss constraint $\tilde{\mathcal G}_{\mathbf i} = 0$, emerges in Γ_A. This system of constraints is

$$\tilde{\mathcal{S}} := -\frac{1}{2\kappa}\frac{1}{\sqrt{|\det(\tilde{E})|}}\,F^{\mathbf{i}\mathbf{j}}{}_{ab}\tilde{E}^a_{\mathbf{i}}\,\tilde{E}^b_{\mathbf{j}} = 0, \tag{6.2.s}$$

$$\tilde{\mathcal{V}}_b := -\frac{1}{\kappa}\,F^{\mathbf{i}}{}_{ab}\,\tilde{E}^a_{\mathbf{i}} = 0, \tag{6.2.v}$$

$$\tilde{\mathcal{G}}_{\mathbf{i}} := -\frac{1}{\kappa}\,\mathcal{D}_a\,\tilde{E}^a_{\mathbf{i}} = 0; \tag{6.2.g}$$

where, on the domain of $\mathtt{A}$ (i.e. on the 'ADM-sector'), the constraints $\tilde{\mathcal{S}}$ and $-\mathrm{i}\tilde{\mathcal{V}}_a$ are precisely the pull backs to Γ_A of $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{C}}_a$ along $\mathtt{A}$, respectively. The corresponding constraint functions on Γ_A are $C[N,N^a,N^{\mathbf{i}}] := \int_\Sigma(\tilde{\mathcal{S}}N + (\tilde{\mathcal{V}}_a - \tilde{\mathcal{G}}_{\mathbf{i}}\,A^{\mathbf{i}}_a)N^a + \tilde{\mathcal{G}}_{\mathbf{i}}\,N^{\mathbf{i}})\mathrm{d}^3x$, where N, N^a and $N^{\mathbf{i}}$ are arbitrary real valued smearing fields on Σ. However, (6.2) on Γ_A defines only the constraint system for the *complex* rather than the real, Lorentzian general relativity. To recover the latter, the so-called reality conditions, a further constraint, have to be imposed (see [9]). One of these conditions is $\mathrm{Im}\,(\tilde{E}^a_{\mathbf{i}}\tilde{E}^b_{\mathbf{j}}\eta^{\mathbf{i}\mathbf{j}}) = 0$, which, together with the implicit assumption $\det(\tilde{E}) \neq 0$ that we had in (6.2.s), implies that $q^{ab} := |\det(\tilde{E})|^{-1}\tilde{E}^a_{\mathbf{i}}\tilde{E}^b_{\mathbf{j}}\eta^{\mathbf{i}\mathbf{j}}$ is nondegenerate and negative definite, as it is on the ADM-sector. The other part of the reality conditions will not be used in what follows.

The spinor Chern–Simons functional Y will now be a function of the configuration variable $A^{\mathbf{i}}_a$ alone, and its functional derivative is $\delta Y/\delta A^{\mathbf{i}}_a = F_{\mathbf{i}\,bc}\epsilon^{bca}$, where ϵ^{abc} is the alternating Levi-Civita *symbol*. (In fact, Y is just one-fourth the Chern–Simons functional $Y_{(2,0)}$ built from the connection $A^{\mathbf{i}}_a$ on $\mathbf{S}^{(AB)}(\Sigma)$.) In the symplectic formalism its diffeomorphism- and gauge invariance are expressed by $\{C[0,N^a,0],Y\} = 0$ and $\{C[0,0,N^{\mathbf{i}}],Y\} = 0$, respectively, which can also be verified directly (using the Bianchi identity for $F^{\mathbf{i}}{}_{ab}$ in the latter case). Its Poisson bracket with $-C[N,0,0]$ coincides with $\dot{Y}$ given by (3.2), provided the constraints are satisfied. Thus in the generic case the Hamiltonian vector field of Y is not tangent to the constraint surface.

On the ADM-sector the conformal rescaling of the previous section yields the transformation $\tilde{E}^a_{\mathbf{i}} \mapsto \Omega^2\,\tilde{E}^a_{\mathbf{i}}$ and $A^{\mathbf{i}}_a \mapsto A^{\mathbf{i}}_a - \Omega^{-1}(\varepsilon^{\mathbf{i}\mathbf{j}}{}_{\mathbf{k}}\,E^b_{\mathbf{j}}D_b\Omega + \mathrm{i}\delta^{\mathbf{i}}_{\mathbf{k}}\,\dot{\Omega})\vartheta^{\mathbf{k}}_a$, and hence we define the conformal rescaling on the whole Γ_A by the same formulae. In terms of the basic variables $E^a_{\mathbf{i}}$ is defined by $\tilde{E}^a_{\mathbf{i}} =: \sqrt{|\det(\tilde{E})|}E^a_{\mathbf{i}}$, and $\vartheta^{\mathbf{i}}_a$ is the dual of $E^a_{\mathbf{i}}$. Thus the vector field on Γ_A corresponding to the infinitesimal conformal rescaling by $(\delta\Omega,\delta\dot{\Omega})$ is $\mathcal{K} = (-\varepsilon^{\mathbf{i}\mathbf{j}}{}_{\mathbf{k}}\,E^b_{\mathbf{j}}(D_b\delta\Omega)\vartheta^{\mathbf{k}}_a - \mathrm{i}\delta\dot{\Omega}\vartheta^{\mathbf{i}}_a, 2\delta\Omega\,\tilde{E}^a_{\mathbf{i}})$. Then $\mathcal{K}(Y) = 2\mathrm{i}\kappa C[\delta\dot{\Omega}, -\mathrm{i}D^a\delta\Omega, \mathrm{i}A^{\mathbf{i}}_aD^a\delta\Omega]$. $\mathcal{K}$ is a Hamiltonian vector field only if $\delta\Omega = 0$, whenever the corresponding generator function is $\Phi = -\frac{2}{\kappa}V[\delta\dot{\Omega}]$, where $V[n] := \int_\Sigma n\sqrt{|\det(\tilde{E})|}\mathrm{d}^3x$. *Thus the scalar constraint $\tilde{\mathcal{S}}$ smeared by n is just the Poisson bracket of $V[n]$, a functional of the momenta alone, and the spinor Chern–Simons functional Y, a functional of the configuration variable only; and $\tilde{\mathcal{S}} = 0$ is a consequence of the requirement of the invariance of the spinor (or Ashtekar-) Chern–Simons functional with respect to infinitesimal spacetime conformal rescalings.* Since in the Ashtekar formulation $A^{\mathbf{i}}_a$ does not have a metric content, *it is only $V[n]$ (i.e. the physical 3-volume or Misner's time if the reality conditions are satisfied) through which the spatial metric enters the dynamics.*

$T[f]$, the smeared version of York's local time, can be considered as a function on Γ_A, too, and its Poisson bracket with Y is $\{Y,T[f]\} = -\frac{1}{3}\mathrm{i}\int_\Sigma fF_{\mathbf{i}\,[ab}(\Gamma^{\mathbf{i}}_{c]} - A^{\mathbf{i}}_{c]}) - \frac{2}{3}\mathrm{i}\kappa\int_\Sigma\tilde{\mathcal{V}}_aE^a_{\mathbf{i}}\eta^{\mathbf{i}\mathbf{j}}\,E^b_{\mathbf{j}}D_bf\,\mathrm{d}^3x$, which is precisely (5.3), where $\Gamma^{\mathbf{i}}_a := \frac{1}{2}\varepsilon^{\mathbf{i}}{}_{\mathbf{j}}{}^{\mathbf{k}}\,\vartheta^{\mathbf{j}}_bD_aE^b_{\mathbf{k}}$, the connection 1-form of the Levi-Civita connection determined by $E^a_{\mathbf{i}}$ as an orthonormal basis. Interestingly enough, a similar formula for the vector constraint can be obtained by means of $M[f] := \int_\Sigma f\Gamma^{\mathbf{i}}_a\tilde{E}^a_{\mathbf{i}}\mathrm{d}^3x$, which is a (gauge dependent) function of the*momentum variable only*: $\{Y,M[f]\} = -\mathrm{i}\kappa\int_\Sigma fF_{\mathbf{i}\,[ab}\Gamma^{\mathbf{i}}_{c]} - \mathrm{i}\kappa^2\int_\Sigma\tilde{\mathcal{V}}_aE^a_{\mathbf{i}}\eta^{\mathbf{i}\mathbf{j}}\,E^b_{\mathbf{j}}D_bf\,\mathrm{d}^3x$. However, it is not clear whether the diffeomorphism and Gauss law constraints themselves can be recovered as the pure Poisson bracket of (the diffeomorphism and gauge invariant) Y and some functionally differentiable functions $D,G:\Gamma_A \to \mathbf{C}$, too, or Y generates only the constraint for the time evolution, but not for the kinematical symmetries.

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